Contents

Chapter 1

INTRODUCTION

This are notes which I have prepared for the lectures to be given to the students preparing for IIT-JEE.

Chapter 2

Determinants and Matrices

2.1 Consistency and Determinancy of a linear system

A system of linear equation is as follows

$$\sum_{j=1}^{n} a_{ij}x_j = d_i$$

where $j \in \{1, 2, ..., n\}$

Definition 2.1.1. Homogenous & nonhomogenous linear system

If the terms d_i's are all zero then the system is said to be a *homogenous system*. If the system has atleast one d_i nonzero then the system is said to be *non-homgenous system*.

Depending on whether a system has atleast one solution or no solution it is defined as follows

Definition 2.1.2. Consistent sytem of linear equations

A linear system is defined as *consistent* sytem if it has **atleast** one solution. If a system has no solution then the system is said to be *inconsistent*.

Definition 2.1.3. Determinate or Indeterminate linear system

A system is defined as a *Determinate system* if it has a unique solution if it has more than one solution then it is said to be *indeterminate*.

Definition 2.1.4. Trivial solution If a system has $x = 0, y = 0, z = 0$ as the only solution then the system is said to have trivial solution. The system is said to have nontrivial solution if there are infinite solutions.

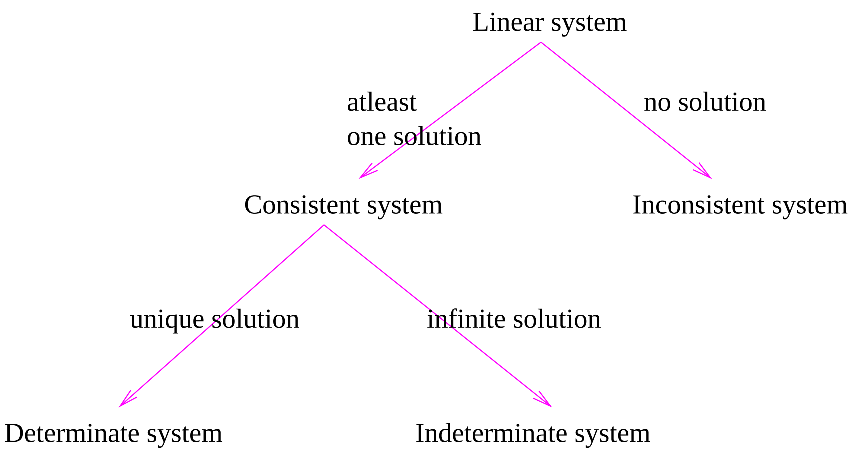


Figure 2.1: Linear system solutions.

We will discuss the above terms only for system of two and three unknowns only. We would learn the following things with analogy from 3D geometry.

1. **Homogenous system of two unknown.**

$$a_1x + b_1y = 0$$
$$a_2x + b_2y = 0$$

Let us use coordinate geometry to understand this system. These equations are lines passing through the origin. Hence their obvious solution that is point of intersection is origin,$(0, 0)$. *Now there are two possibilities,*
1. The lines are parallel to each other, means they are lying on one another.
2. The lines are not parallel to each other, means $(0, 0)$ *is the only intersection.*
Now for the lines to be parallel we need their slopes to be equal, $-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$*, this is nothing but the* $\Delta = 0$ *condition. That's why the determinant of the coefficients of* x *and* y *would play a vital role in deciding the nature of the system.* See the next figure.

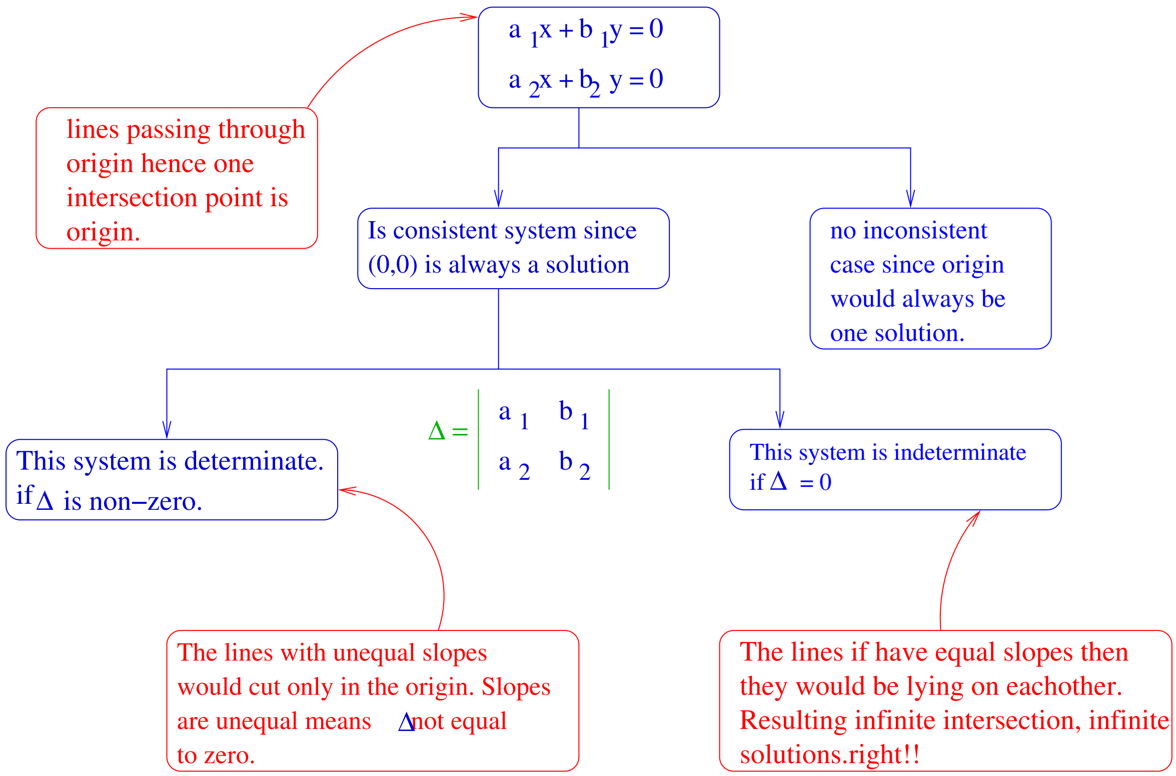


Figure 2.2: Homogenous linear system in two unknowns.

2. **Non-homogenous sytem of two unknowns.**

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

where atleast one of c_1, c_2 is non-zero.
Let us make a note of what the following terms mean in two-Dim geometry.

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Again with analogy from coordinate geometry, these lines represent two lines, atleast one of which is not passing through the origin.

Now there are three cases:

1 The lines are parallel to each other but are away at a constant non-zero distance from each other.

This is for e.g the lines $x + y = 1$ *and* $x + y = 2$ *are parallel at constant distance* $\frac{1}{\sqrt{2}}$ *from each other.*

2. The lines are parallel to each other and are one above the other, i.e the distance between them is zero.

This is for e.g the lines $x + y = 1$ *and* $2x + 2y = 2$ *which are parallel and at the same time lying on each other.*

3. The lines are not parallel that means they got to intersect only in one single point(unique solution)

This is in for e.g the lines $x + y = 1$ *and* $x - y = 1$ *which are not parellel and hence have to intersect in a unique point.*

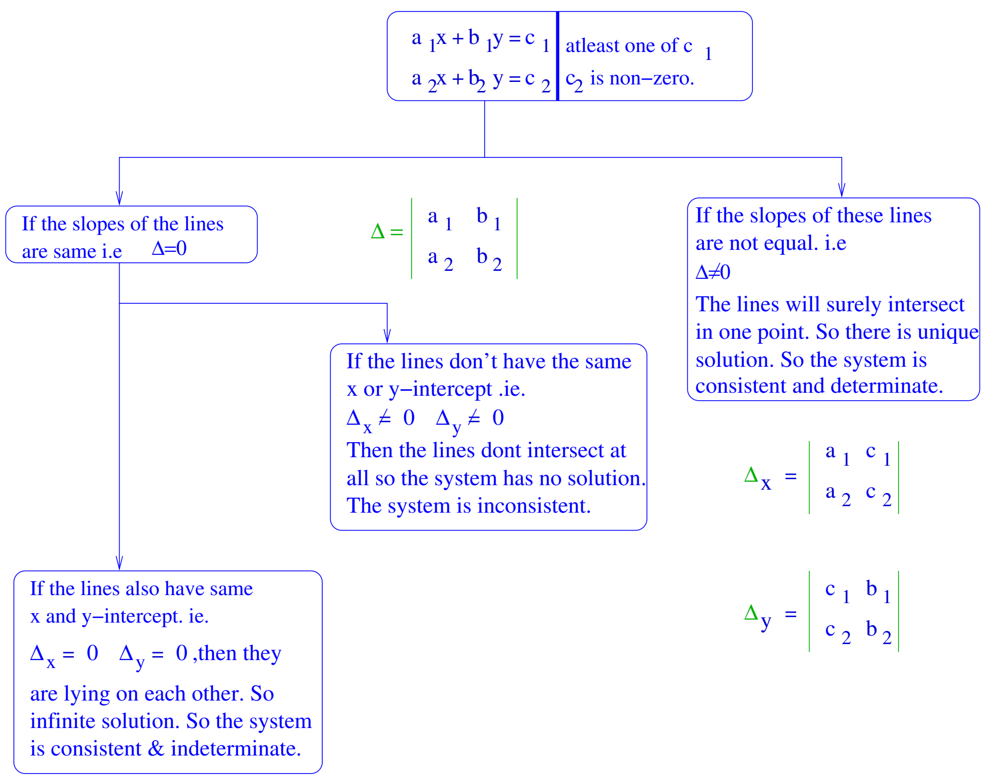


Figure 2.3: Non-homogenous linear system in two unknowns.

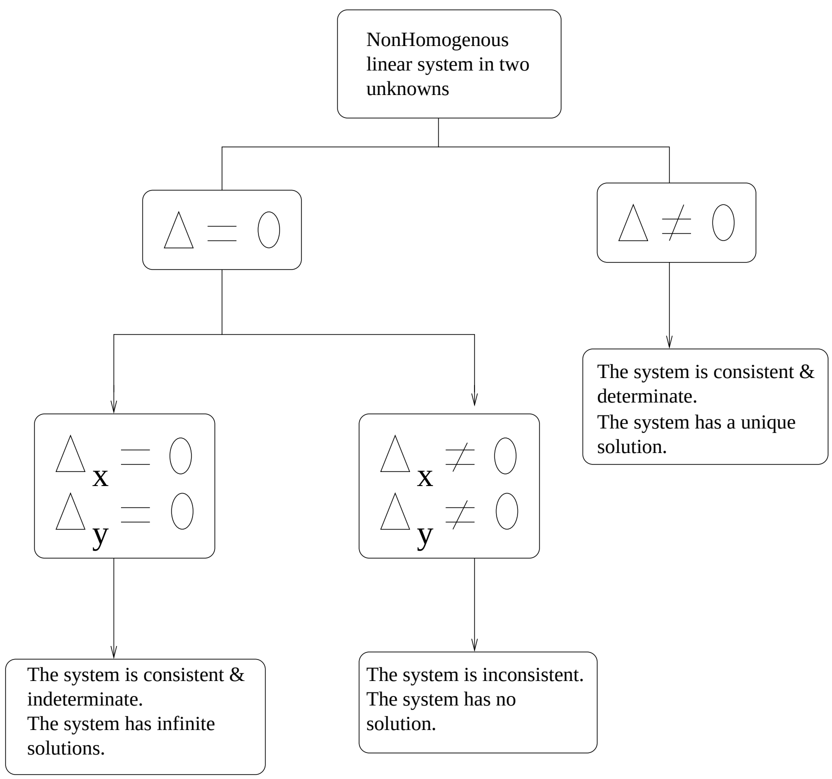


Figure 2.4: Conclusion of the above discussion on the basis of Δ.

Chapter 3

IIT 2004 Mathematics Paper

3.1 Mains paper

(a) Find the center and radius of the circle determined by all complex numbers $z = x + iy$ satisfying $\left|\frac{z-\alpha}{z-\beta}\right| = k$, where $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$ are fixed complex numbers and $k \neq 1$.
solution : The centre is at $\frac{\alpha-k^2\beta}{1-k^2}$ i.e., at $\left(\frac{\alpha_1-k^2\beta_1}{1-k^2}, \frac{\alpha_2-k^2\beta_2}{1-k^2}\right)$ and radius is $k\left|\frac{\alpha-\beta}{1-k^2}\right|$.

(b)

Chapter 4

Limits And Continuity

4.1 Introduction to Limits and Continuity

Why do we at all need to learn the concept of limit and continuity ?

Let me begin with a story. A road was thought to be constructed through a jungle joining two villages on either side of the jungle. But then there was a problem, through the centre of the jungle there was a river passing. Now the villagers from both the villages had two problems.

(a) They had to built the road from either side to reach at the same place opposite about the river.

(b) Then after getting the road to come opposite to each other they needed to construct the bridge over the river.

Now the success of the road linking both villages requires (necessary condition) both the roads to reach exactly opposite on the river. If this is met then the next stage of constructing the bridge will continue. So here the basic requirement is getting the road on exact opposite bank.

Now why this whole story ! This story goes in parellel with the concept of continuity. The concept of continuity means constructing a continuous road linking both the villages. So for continuity, the necessary condition is existence of limit that is same as constructing the road on both the sides of river to come at the same location opposite of the river. Now the next step is filling the gap that is building the bridge.

4.2 Limits

We see

$\lim_{x\to a} f(x) = l$ (4.1)

then don't have misunderstanding due to that equality sign. There is no equality when we specify a limit to $f(x)$ at $x = a$. Let us understand what do we mean by the lim specified above. $\lim_{x\to a} f(x) = l$ actually means $f(x) \to l$ as $x \to a$. Since in our scope we have the x tending to a in one dimension we only have to see how it tends to a from the left of a and right of a. So we need to find how we find $\lim_{x\to a^-} f(x)$ i.e the L.H.L (left hand limit) and $\lim_{x\to a^+} f(x)$ i.e R.H.L (right hand limit). To be true there is nothing like equation

If a, b, c are real numbers then

(a) If $a < b \Rightarrow a + c < b + c$ (note its true whether $c > 0$ or $c < 0$ or $c = 0$

(b) If $a < b \Rightarrow ac < bc$ for $c > 0$ or $ac > bc$ for $c < 0$.

(c) If $a < b$ and $p, q > 0$ then $a^p < b^p$ and $a^{1/q} < b^{1/q}$.

(d) If $0 < a < 1 \Rightarrow 0 < \cdots < a^3 < a^2 < a < 1$
and if $a > 1 \Rightarrow 1 < a < a^2 < \cdots < a^n - 1 < \cdots < \infty$.

4.3 A.M-G.M inequality

There is this heavily used inequality called the **A.M-G.M inequality** If $a, b > 0$ then $\frac{a+b}{2} \geq \sqrt{ab}$
It can also be generalised for n **positive** real numbers $\{a_1, a_2, ..., a_n\}$ then we have $\frac{\sum a_i}{n} \geq (\Pi a_i)^{1/n}$
NOTE A.M-G.M inequality works only for positive real values.

And A.M-G.M inequality will be useful when you see symmetry in the question since the inequality is also symmetrical or cyclic.. Can you prove the A.M-G.M inequality with two methods.
Hint: Method-1 is to use $(a-b)^2 \geq 0$ and Method-2 is to use

4.4 Symmetry

The maximum of minimum value (which ever exists) of an expression is attained only when all the terms in the symmetrical expression are equal.

Example 4.4.1. The minimum value of the expression $(a-b)^2$ exists and is 0 when $a = b$.

Chapter 5

Indefinite Integration

5.1 Basic Formulae from definition of Indefinite integration

Indefinite integration is inverse operation of differentiation. If $\frac{d}{dx}F(x) = f(x)$ then $\int f(x)\,dx = F(x)$. So our definition of Indefinite integral goes as

Definition 5.1.1. Indefinite integral is defined as antiderivatives.

(a) $\frac{d}{dx}x^{n+1} = (n+1)x^n \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

(b) $\frac{d}{dx}ln(x) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = ln(x) + c$

(c) $\frac{d}{dx}a^x = a^x lna \Rightarrow \int a^x\,dx = \frac{a^x}{lna} + c$

(d) $\frac{d}{dx}e^x = e^x \Rightarrow \int e^x\,dx = e^x + c$

(e) $\frac{d}{dx}\sin(x) = \cos(x) \Rightarrow \int \cos(x)\,dx = \sin(x) + c$

(f) $\frac{d}{dx}\cos(x) = -\sin(x) \Rightarrow \int \sin(x)\,dx = -\cos(x) + c$

(g) $\frac{d}{dx}\tan(x) = \sec^2(x) \Rightarrow \int \sec^2(x)\,dx = \tan(x) + c$

(h) $\frac{d}{dx}\cot(x) = -\text{cosec}^2(x) \Rightarrow \int \text{cosec}^2(x)\,dx = -\cot(x) + c$

(i) $\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \Rightarrow \int \sec(x)\tan(x)\,dx = \sec(x) + c$

(j) $\frac{d}{dx}\text{cosec}(x) = -\text{cosec}(x)\cot(x) \Rightarrow \int \text{cosec}(x)\cot(x)\,dx = -\text{cosec}(x) + c$

Now these basic formulae (as we will be calling them) are the only formulae due to definition available to us for solving problems. To

solve any problem in integration we have to reduce the problem to one of the above formulae.
Now the obvious first thing that comes to our mind is, what is that c in the indefinite integration formulae above. It is called the constant of integration. Why should we need it ? well that is what u have to do (think!) make use of definition

Theorems for integration which can be derived from the definition.

(a) $\int kf(x)\,dx = k\int f(x)\,dx$

(b) $\int (f(x)\pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ this can be extended to integral of sum of a finite number of functions.

5.2 Decomposition of an integrand as sum of integrands

This is the basic method of solving integral problems. We have to decompose an integrand into two or more integrands and use the second theorem of adding the individual finite number of integrals.

Example 5.2.1. $\int \sec^2(x)\mathrm{cosec}^2(x)\,dx$
use

$$\sec^2(x)\mathrm{cosec}^2(x) = \sec^2(x) + \mathrm{cosec}^2(x)$$

$= \int \sec^2(x)\,dx + \int \mathrm{cosec}^2(x)\,dx = \tan(x) - \cot(x) + c$

5.3 Method of Substitution

Method of substitution is helpful in simplifying an integration problem to one of the simpler forms where the above basic formulae can be applied to solve the problem. First let us see the theory.
If $\int f(x)\,dx = F(x)$ and x depends on another variable t so $x = \phi(t)$ so

$$dx = \phi'(t)dt$$

now

$$\int f(x)\,dx = \int f(\phi(t))\phi'(t)dt$$

Note the LHS integral may not be easy to solve or may not be in the above listed basic formulae but the RHS is most of the times in simpler form and using basic formulae we solve it.

Let us see with an example. We know integrals of only $\sin(x)$ and $\cos(x)$. Now we see of $\tan(x)$.

Example 5.3.1.

$$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx$$

Now we use method of substitution. put $t = \cos(x) \Rightarrow dt = -\sin(x)dx$

$$= -\int \frac{dt}{t} = -ln|t| + c = -ln|\cos(x)| + c = ln|\sec(x)| + c$$

Let us come out of integration problem solving for a moment. Let us move freely in other parts of basic Calculus. Now in this method of substitution what are we exactly doing. We were working in the space of variable x and by the substitution we go to a new space of variable t. In this new space do we go alone? No we take our problem of integration to be solved. Now the structure, appearance of our problem has changed in this new space. You have done this kind of a thing before also, do u remember? In logarithms, you had to multiply two numbers say, you apply the log function and carry them there the numbers altogether change and multiplication becomes addition you add and then what do u do? You apply the antilog, which is nothing but the inverse function of log function (i.e. exponential function) and get back to the original space of x. This is why we are more interested in those functions which have inverse. A function can simplify my problem by traversing in a different space but if there is no inverse function to that function then I can't get back the solution to my original space. This is the significance of inverse function.
Coming back to integration.
Similarly by the same approach as in the example above, we have

Example 5.3.2.

$$\int \cot(x)\,dx = ln|\sin(x)| + c$$

Now we generalize an obsevation from this two formulae. That is, in both the problems we are getting the derivative of the denominator in the numerator of the integrand. For a function $f(x)$

$$\int \frac{f'(x)}{f(x)}\,dx = ln|f(x)| + c$$

Note that this is the most frequently encountered formulae. You have to *observe* whether the derivative of the denominator is there in the numerator or not!.

Example 5.3.3.

$$\int \sin^{10}(x)\cos(x)\,dx = \int \sin^{10}(x)\,d(\sin(x))$$

put $t = \sin(x)$ we get,

$$= \int t^{10}\,dt = \frac{t^{11}}{11} + c = \frac{\sin^{11}(x)}{11} + c$$

From the above example we can generalize,

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

Now consider the following

$$(e^x f(x))' = e^x f'(x) + e^x f(x)$$

using product rule in differentiation and integrating with respect to (w.r.t) x on both sides

$$\int (e^x f(x))'\,dx = \int (e^x f'(x) + e^x f(x))\,dx$$

By definition, integration and differentiation are inverse operations, so we get

$$e^x f(x) = \int (e^x f'(x) + e^x f(x))\, dx$$

So we are with another important formulae,

$$\int e^x (f'(x) + f(x))\, dx = e^x f(x)$$

Note that this formulae is very easy to prove but very difficult to use. A problem would be simplied by this formula but then it is not easily visible that it is being embedded into the problem.

Example 5.3.4. $\int e^{x/2} \sin(\frac{x}{2} + \frac{\pi}{4})\, dx$
Put $t = \frac{x}{2}$ so we get

$$= \sqrt{2} \int e^t (\sin(t) + \cos(t))\, dt = \sqrt{2} e^{x/2} \sin x/2$$

Example 5.3.5. $\int \frac{e^x(1+x\ln(x))}{x}$ solve it yourself.

Example 5.3.6. $\int(\sin(\log(x)) + \cos(\log(x)))$??

Bad way of writing results in bad way of thinking

If $\int f(x)\, dx = F(x)$ then can we say that

$$\int f(ax+b)\, dx = \frac{F(ax+b)}{a}$$

Yes! that is right and the right way of writing the result. Shall we prove that ?
put $t = ax + b \Rightarrow dt = adx$ then we have

$$\int f(ax+b)\, dx = \frac{1}{a} \int f(t)\, dt = F(t) + c$$

But the wrong way this is written or taught is
If $\int f(x)\, dx = F(x)$ then $\int f(ax+b)\, dx = \frac{F(ax+b)}{\frac{d}{dx}(ax+b)}$ This the mistake students do when they write or think in the above wrong way, they

write If$\int f(x)\,dx = F(x)$ then $\int f(ax^2+bx+c)\,dx = \frac{F(ax^2+bx+c}{\frac{d}{dx}(ax^2+bx+c)}$ This is absolutely wrong. That second way of writing the integral when x is replaced by $ax+b$ is the real culprit. So we should not forget that the result for linear substitution is got by *method of substitution*. We should be away from generalizing results till we prove them.

(a) List of Nine formulae useful for solving problems

i. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$

ii. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}ln|\frac{x-a}{x+a}| + c$

iii. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}ln|\frac{a+x}{a-x}| + c$ (multiply by -1 in equation 2 above.)

iv. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = ln|x+\sqrt{x^2+a^2}| + c$

v. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = ln|x+\sqrt{x^2-a^2}| + c$

vi. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a}) + c$

vii. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}ln|x+\sqrt{x^2+a^2}| + c$

viii. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}ln|x+\sqrt{x^2-a^2}| + c$

ix. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(\frac{x}{a}) + c$

(b) To solve the integrals of the form

$$\int \frac{1}{ax^2+bx+c}\,dx, \int \frac{1}{\sqrt{ax^2+bx+c}}\,dx, \int \sqrt{ax^2+bx+c}\,dx$$

You have to first get ax^2+bx+c in one of the forms, x^2+a^2, x^2-a^2 or a^2-x^2. Then use one of the above formulae to solve the integral.

(c) To solve the integrals of the form

$$\int \frac{px+q}{ax^2+bx+c}\,dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx, \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

write

$$px+q = m\frac{d}{dx}(ax^2+bx+c) + n$$

evaluate the constants p and q and solve them.

Remember $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}$ and $\int f'(x)\sqrt{f(x)}\,dx = \frac{2}{3}[f(x)]^{3/2}$

(d) To solve

$$\int \frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$$

the substitution is $px+q = 1/t$ reduces the problem to the previously discussed forms containing quadratic. Same substitution works for

$$\int \frac{dx}{(px+q)^k\sqrt{(ax^2+bx+c)}}$$

(a) To solve the integrals of the form

$$\int R(\sin(x), \cos(x))\, dx$$

put

$$t = \tan(x/2) \Rightarrow dx = \frac{2dt}{1+t^2}, \sin(x) = \frac{2t}{1+t^2}, \cos(x) = \frac{1-t^2}{1+t^2}$$

and

$$\tan(x) = \frac{2t}{1-t^2}$$

(b) If $R(\sin(x), \cos(x)) = R(-\sin(x), -\cos(x))$ then put $t = \tan(x)$ But before substituting divide numerator and denominator of the integrand with $\cos^2(x)$ so that you provide dx with $\sec^2(x)$ required.

(c) To solve the integrals of the form

$$\int \frac{a\cos(x) + b\sin(x)}{p\cos(x) + q\sin(x)}\, dx$$

write

Numerator $= m\frac{d}{dx}$ (denominator) $+n$(denominator)

(d) To solve the integrals of the form

$$\int \frac{a\cos(x) + b\sin(x) + c}{p\cos(x) + q\sin(x) + r}\, dx$$

write

Numerator $= m\frac{d}{dx}$(denominator) $+n$(denominator)$+l$

5.4 Partial Fractions

We can better learn this by examples

Example 5.4.1.

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$\frac{1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$\frac{2x^2+1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$$\frac{3x+1}{(x^2+1)^2(x+1)^2} = \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x^2+1)^2} + \frac{C_1}{x+1} + \frac{C_2}{(x+1)^2}$$

Observations from the above examples

(a) All the integrands on the LHS. are proper rational function (i.e. deg(Numerator function) $<$ deg(Denominator function) If they are not then you should divide the Numerator with the denominator and then get the proper rational function.

(b) Nextly the degree of the numerator of every RHS term is *one* less than the denominator.

(c) Third if a factor occurs in the denominator with a multiplicity then that many times the partial fraction corresponding to that factor is repeated increasing its power by one till that multiplicity.

5.5 Integration by parts

There are two forms,

(a) If $f(x)$ and $g(x)$ are two continuous functions then

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int (\frac{d}{dx}f(x)\cdot \int g(x)\,dx)\,dx$$

Now integration by parts doesn't work for product of any two functions (or else would have been a theorem) but works with few restrictions the functions f and g have to be chosen in such a way that the process of solving simplifies. For this to happen we would choose the functions f and g such that it follows **ILATE** order.(where ILATE stands for inverse, log, algebraic, trigonometric and exponential functions respectively).

(b) The other way is we have two functions of x, u and v then

$$(u\cdot v)' = udv + vdu$$

so we can write this as

$$\int u\,dv = uv - \int v\,du$$

5.6 Important formulae and observations

(a) $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$

(b) $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) = \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$

(c) $1+\cos(\theta) = 2\cos^2(\theta/2)$ and $1-\cos(\theta) = 2\sin^2(\theta/2)$

(d) $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$ and $\tan(\frac{\pi}{4}\pm\theta) = \frac{1\pm\tan(\theta)}{1\mp\tan(\theta)}$

(e) $x^2 + \frac{1}{x^2} = (x+\frac{1}{x})^2 - 2 = (x-\frac{1}{x})^2 + 2$
$\frac{d}{dx}(x+\frac{1}{x}) = 1 - \frac{1}{x^2}$ and $\frac{d}{dx}(x-\frac{1}{x}) = 1 + \frac{1}{x^2}$

(f) $\sec^2(x)\mathrm{cosec}^2(x) = \sec^2(x) + \mathrm{cosec}^2(x)$

5.7 Important Points

(a) $\log(x)$ function is defined only for positive values of x. In differentiation, we write

$$\frac{d}{dx}\log(x) = \frac{1}{x}$$

here we should write

$\dfrac{d}{dx}\log|x| = \dfrac{1}{x}$ (How?? Use definition of modulus function)

here x is any real number (positive or negative). So that by definition of indefinite integration we can write (properly) $\int \frac{1}{x}\,dx = \log|x| + c$

(b)

5.8 Good Problems

(a) $\int \sqrt{\frac{1-x}{1+x}}\,dx$ put $x = \cos(2\theta)$

(b) $\int \sqrt{\frac{x-1}{x+1}}\frac{dx}{x^2}$ in this problem we can observe x^2 in the denominator and we know that $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$ so we substitute first $t = \frac{1}{x}$ and then we have the form of the first problem above.

(c) $\int \frac{\sqrt{\tan(x)}}{\sin(x)\cos(x)}\,dx$ see that $R(\sin(x), \cos(x)) = R(-\sin(x), -\cos(x))$ so the substitution that works is $t = \tan(x)$. But before making the substitution you should simplify the integrand a bit. For this substitution divide by $\sec^2(x)$ both numerator and denominator and wherever required use $\sec^2(x) = 1 + \tan^2(x)$

(d) $\int \sin(\log(x))\,dx$
Sometimes converting the $\log(x)$ function in exponential function form simplifies the problem. Here put $t = \log(x) \rightarrow x = e^t \rightarrow dx = e^t dt$
$= \int \sin(t)(e^t)\,dt$
This can be simplified with integration by parts.
Another example of this tactics is

$$\int \sin(\log(x)) + \cos(\log(x))\,dx$$

and is simplified by converting the integrand from logarithmic form to exponential form by substituting $t = \log(x)$.

(e) $\int xsin(2\log(x))\, dx$
Note this problem has $2\log(x)$ as a parameter for $\sin x$ function, which will cause problem in getting us the integral in simplified elementary form. So we got to substitute $2\sin(x)$ as t. And proceed to get the integral in form of integration by parts.

(f) $\int \frac{dx}{x(x^2+1)^3}$
Here we can directly employ partial fraction but if you observe then you can simplify the integrand more.
Put $t = x^2 + 1$ so the integral becomes,
$\int \frac{dt}{(t-1)t^3} = \int \frac{A}{t-1}\, dt + \int \frac{B}{t}\, dt + \int \frac{C}{t^2}\, dt + \int \frac{D}{t^3}\, dt$

(g) In $\int (ax+b)^{1/n}\, dx$ and $\int (\frac{ax+b}{cx+d})^{1/n}\, dx$
put $t^n = ax + b$ and $t^n = (\frac{ax+b}{cx+d})$ respectively.
Solve $\int \frac{x}{(2+3x)^{1/3}}\, dx$
put $t^3 = 2 + 3x$

(h) $\int \sqrt{(\frac{1-x}{1+x})}\frac{dx}{x}$.
put $t^2 = \frac{1-x}{1+x} \Rightarrow x = \frac{1-t^2}{1+t^2}$ differentiate and getting the integral in variable we can solve the new integral by partial fractions.
Can you make any other substitution try put $x = \cos(2\theta)$

(i) For a function f to be integrable,

i. f is continuous.
ii. f is discontinuous.
iii. f is discontinuous at finite number of points in the domain should work.
iv. f is discontinuous at infinite number of points should work.
v. f f is differential function.

For a function f to be integrable we need that the function f to be continuous. Now if there are finite number of points where the function is not continuous then we will break the domain at this points of discontinuites and then in this broken each small domain the function is continuous and hence integrable. Now if the function is differentiable then it is also continuous. But if the function is known to be discontinuous at infinite number of points then the function is not integrable.

(j) Is the function $\int |x|\, dx$ continuous and further is this function differential also state the reason justifying your assertion.
The function $|x|$ is continuous on the real domain. So the function is integrable. Now integration smothens every integrable function, that means every integrable function may not be differentiable but its integral is differentiable. Here

$$|x| = x \Rightarrow \int |x|\, dx = \int x\, dx = \frac{x^2}{2}, x \geq 0$$

$$|x| = -x \Rightarrow \int |x|\, dx = \int -x\, dx = -\frac{x^2}{2}, x < 0$$

hence we have that

$$\int |x|, dx = \begin{cases} \frac{x^2}{2} & \text{if } x \geq 0 \\ -\frac{x^2}{2} & \text{if } x < 0 \end{cases}$$

(k) $\int \frac{x^2+1}{1+x^4}\, dx, \int \frac{x^2-1}{1+x^4}\, dx, \int \frac{x^2}{1+x^4}\, dx, \int \frac{1}{1+x^4}$
In this problem we are actually using the following formulae

$$d(x + \frac{1}{x}) = 1 - \frac{1}{x^2}$$
$$d(x - \frac{1}{x}) = 1 + \frac{1}{x^2}$$

Then we use

$$(x^2 + \frac{1}{x^2}) = (x + \frac{1}{x})^2 - 2 = (x - \frac{1}{x})^2 + 2$$

Chapter 6

Mathematics Basics

6.1 Set theory

All branches of mathematics were developing independent of each other in the beginning but then arised the need to unify all these branches under common name *mathematics*. Intially some mathematicians claimed that logic can be said to be genesis of the different branches of mathematics studied at that time. But then Set Theory was being proved to be the genesis of Mathematics.

Definition 6.1.1. Set
A set is a collection of *distinct* objects.

Definition 6.1.2. Equal Sets
Two sets A and B are called *equal sets* if

Definition 6.1.3. Subset
A is said to be a subset of the set B if every element in A is contained in B. Now note here that the sets A and B may also be equal under this definition.Denoted as $A \subset B$
Proper Subset : A set A is said to be a *proper subset* of set B if all the elements of set A are contained in the set B and there is atleast one element of set B that is not contained by A.

Definition 6.1.4. Null set
A set is said to be *null* if it contains no element, denoted as ϕ.

Definition 6.1.5. Disjoint set
Two sets A and B are said to be disjoint if no element of set A is in set B.

VENN DIAGRAMS

Operations on sets
The operations working on sets are *union*,*intersection*,*complement*.

(a) **Union**: Union of two sets A and B is the set of all elements from set A and set B.
(b) **Intersection**: Intersection of two sets A and B is the set containing common elements of A and B.
(c) **Complement**:Complement of a set A is the set of elements which are not in A(then where are these elements from) but in the U universal set.

Difference of two sets: Difference denoted as $A - B$ is the set of all elements of A not in B.(It is like removing all the elements of B from set A).
$A - B = A \cap B^c$

Properties of these operations on Sets
These properties are called commutativity.(you can see symmetry in the venn diagrams the first two properties.)

(a) $A \cup B = B \cup A$
(b) $A \cap B = B \cap A$
(c) $A - B \neq B - A$

These property is called distributive property.

(a) $(A \cup B) \cup C = A \cup (B \cup C)$
(b) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

These property is called Associative property.

(a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $(A \cap B) \cap C = A \cap (B \cap C)$

Demorgan's Law (the first two formulae are having complement with respect to the Universal set. In the next two the complement is with respect to the set X.

(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$

(c) $X - (A \cup B) = (X - A) \cap (X - B)$

(d) $X - (A \cap B) = (X - A) \cup (X - B)$

Are these statements true, Check?

(a) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(b) $(A - B) = A - (A \cap B)$ what is $(B - A)$ Is it symmetric. Flip the venn diagram once and check??

SET OF NUMBERS

(a) *Natural Numbers*(N):A set $\{1, 2, 3, ...\}$ is called the set of Natural numbers.

(b) *Whole Numbers* : A set $\{0, 1, 2, 3, ...\}$ is called the set of Whole numbers.

(c) *Integers*(Z or I): A set $\{... - 2, -1, 0, 1, 2, ...\}$ is called the set of Integers.

(d) *Rational Numbers*(Q): A set $\{\frac{p}{q} : p, q \epsilon Z$ and $q \neq 0\}$ is called the set of Rational numbers.

(e) *Irrational Numbers* : A set of numbers which are not *Rational.*

(f) *Real Numbers*(R): A set of union of the Rational and Irrational numbers taken together.

(g) *Complex Numbers*(C): A set $\{a + ib : a, b \epsilon R$ and $i = \sqrt{-1}\}$ is called set of complex numbers.

Therefore we have $N \subset W \subset Z \subset Q \subset R \subset C$. Note $R = Irrational \cup Rational$ Can we say $\{Irrationalnumbers\} \subset R$?

Definition 6.1.6. For decimal form of Real Numbers.
When in decimal form, Rational and Irrational numbers are differed in the way that, rational numbers are *non-terminating* and *recurring* while Irrational numbers are *non-terminating* and *non-recurring*.

Modulus or Absolute value
Now we would explore this as a value but in *calculus* we would see this as a function. Defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The basic confusion of students in the definition of the modulus function is that the negative sign given to x when $x < 0$. This sign is to make the negative number put inside the modulus function to a positive number with the same value.Think!!

Example 6.1.7. Therefore $|2.3| = 2.3$, $|-2| = 2$. So we see that the modulus or absolute function chops of the negative sign.

Properties of modulus value

(a) $|a| \geq 0$
(b) $|a| \geq a$
(c) $|a \cdot b| = |a| \cdot |b|$
(d) $|\frac{a}{b}| = \frac{|a|}{|b|}$
(e) $|a + b| \leq |a| + |b|$
(f) $|a - b| \geq ||a| - |b||$
(g) If $|x| \leq a \Rightarrow -a \leq x \leq a$
(h) If $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$

Order properties of Real numbers

(a) If $a < b$ and $b < c \Rightarrow a < c$
(b) If $a < b$ and $c < b \Rightarrow a < c$ Is this true??

(c) If $a < b$ and $c \in R \Rightarrow a + c < b + c$

(d) If $a < b \Rightarrow \begin{cases} ac < bc & \text{if } c > 0 \\ ac > bc & \text{if } c < 0 \end{cases}$

(e) If $a < b \Rightarrow \begin{cases} \frac{1}{a} > \frac{1}{b} & \text{if } ab > 0 \\ \frac{1}{a} < \frac{1}{b} & \text{if } ab < 0 \end{cases}$

(f) For $0 < |a| < 1 \Rightarrow 0 < ... < a^6 < a^4 < a^2 < 1$

(g) For $|a| > 1 \Rightarrow 1 > a^2 > a^4 > a^6 ... > 0$

(h) If $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$
Can we also say similarly that If $a \cdot b = 1 \Rightarrow a = 1$ or $b = 1$???

(i) If $a^2 + b^2 \Rightarrow a = 0$ *and* $b = 0$ (Note that 'and' what is the need?? both have to be zero simultaneously)

6.2 Logarithms

Definition 6.2.1. $\ln_a x$ is a function whose value is defined for $x > 0$,$a > 0$ and $a \neq 1$

Chapter 7

Definite Integration

7.1 Function Symmetry, Translation and Reflection

Here we will see the behaviour of the function $f(x)$ in the coordinate axes.

(a) The graph of the function $f(x-a)$ is obtained by moving the graph of the function $f(x)$ along the positive x-axis.

(b) The graph of the function $f(x+a)$ is obtained by moving the graph of the function $f(x)$ along the negative x-axis.

(c) The graph of the function $f(x)+a$ can be obtained by moving the graph of the function $f(x)$ along the positive y-axis.

(d) The graph of the function $f(x)-a$ can be obtained by moving the graph of the function $f(x)$ along the negative y-axis.

Function elongation or contraction

(a) For $\alpha > 1$, The graph of the function $f(\alpha x)$ is obtained by contracting the graph of $f(x)$ by α factor along the x-axis.

(b) For $0 < \alpha < 1$, The graph of the function $f(\alpha x)$ is obtained by elongating the graph of $f(x)$ by α factor along the x-axis.

(c) For $\alpha > 1$, The graph of the function $\alpha f(x)$ is obtained by elongating the graph of $f(x)$ by α factor along the y-axis.

(d) For $0 < \alpha < 1$, The graph of the function $\alpha f(x)$ is obtained by contracting the graph of $f(x)$ by α factor along the y-axis.

Function Inversion about X-axis and Y-axis

(a) The graph of the function $\mathbf{f}(-\mathbf{x})$ is the inverted graph of $f(x)$ about the *y-axis*.

(b) The graph of the function $-\mathbf{f}(\mathbf{x})$ is the inverted graph of $f(x)$ about the *x-axis*.

Example 7.1.1. $|\frac{x-1}{x+1}| = \frac{x-1}{x+1}$
Note $|x| = x$ only if $x > 0$. So here in this problem we need $x \neq -1$ for the problem to exist and $\frac{x-1}{x+1} > 0$
$\frac{x-1}{x+1} > 0 \Rightarrow (x-1)(x+1) > 0 \Rightarrow x > 1$ or $x < -1$ So it the solution to the equation is $(-\infty, -1) \cup (1, \infty)$.

Example 7.1.2. $|x^2 - 5x + 6| = -(x^2 - 5x + 6)$ Try this. A graphical thinking is good.

Example 7.1.3. $\int_0^x \sqrt{a^2 - x^2}\, dx$ can you think of the solution in a graphical sense. Hint: The integrand $\sqrt{a^2 - x^2}$ is a part of the circle.

Even And Odd functions and Symmetry

Definition 7.1.4. *Even function*
A function $f(x)$ is called as *even* function if it satisfies $f(-x) = f(x)$.(that is if the negative sign is eaten away by the function given along with the independent variable.)

Example 7.1.5. $\cos(x), e^x + e^{-x}, x^2$

Definition 7.1.6. *Odd function*
A function $f(x)$ is called as *odd* function if it satisfies $f(-x) = -f(x)$(that is the function is not able to swallow the negative sign so it throws out.)

Example 7.1.7. $\sin(x), e^x - e^{-x}, x^3$

NOTE:

(a) Now note that the even functions are symmetric about the y-axis and Odd function are symmetric about the *origin*

(b) The modulus function actually reflects the part of the graph below the x axis under the $x$axis acting as mirror. We will be using this knowlegde to solve problems in definite integration.

Definition 7.1.8. Definite Integration
If $f(x)$ is integrable and has the antiderivative as $F(x)$ that is $F'(x) = f(x)$ in $[a, b]$ then the definite integral with the lower limit as a and upper limit as b is defined as $\int_a^b f(x)\,dx = F(b) - F(a)$ called the *Newton-Leibnitz formula*
NOTE: Newton Leibnitz formula to compute the definite integral of a **continuous** function on $[a, b]$ only if,
$F'(x) = f(x)$ is fulfilled in the whole interval $[a, b]$,i.e. the antiderivative must be a continuous function on the whole interval $[a, b]$. A *discontinuous* anti-derivative would lead to wrong result.

Example 7.1.9. Evaluate $\int_{-1}^{1} \frac{d}{dx}(\tan^{-1}(\frac{1}{x}))\,dx$
$\int_{-1}^{1} d(\tan^{-1}(\frac{1}{x})) = \tan^{-1}(1) - \tan^{-1}(-1) = -\frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$
But note that the function $\tan^{-1}(\frac{1}{x})$ which is the anti-derivative of the integrand above is not continuous on $[-1, 1]$. But we may circumvent this shortcoming of the anti-derivative $\tan^{-1}(\frac{1}{x})$ as follows,
$\int_{-1}^{1} d(\tan^{-1}(\frac{1}{x})) = \int_{-1}^{0} d(\tan^{-1}(\frac{1}{x}) + \int_{0}^{1} d(\tan^{-1}(\frac{1}{x}))$
Now note that on $[-1, 0)$ and $(0, 1]$ the function $\tan^{-1}(\frac{1}{x})$ is continuous. Hence we get,
$\int_{-1}^{0} d(\tan^{-1}(\frac{1}{x})) = \lim_{a\to 0^-} \int_{-1}^{a} d(\tan^{-1}(\frac{1}{x})) = \lim_{a\to 0^-} \tan^{-1}(\frac{1}{a}) - \tan^{-1}(-1) = -\frac{\pi}{2} - (-\frac{\pi}{4}) = -\frac{\pi}{4}$
And

$\int_{0}^{1} d(\tan^{-1}(\frac{1}{x})) = \lim_{a\to 0^+} \int_{a}^{1} d(\tan^{-1}(\frac{1}{x})) = \lim_{a\to 0^+} \tan^{-1}(1) - \tan^{-1}\frac{1}{a}) = \frac{\pi}{4} - (\frac{\pi}{2}) = -\frac{\pi}{4}$
So finally the solution is $-\frac{\pi}{2}$.
Another way of solving this problem,
$\int_{-1}^{1} \frac{d}{dx}(\tan^{-1}(\frac{1}{x}))\,dx = \int_{-1}^{1} \frac{1}{1+\frac{1}{x^2}} \frac{-1}{x^2}\,dx = -\int_{-1}^{1} \frac{1}{1+x^2}\,dx = -(\tan^{-1}(1) - \tan^{-1}(-1) = -(\frac{\pi}{4} - (-\frac{\pi}{4})) = -\frac{\pi}{2}$ Here we have got another anti-derivative of the integrand $\frac{d}{dx}(\tan^{-1}(\frac{1}{x}))$ that is $-\tan^{-1}(x)$ which is continuous on $[-1, 1]$.

Example 7.1.10. Another problem with the same idea is
$\int_0^{\sqrt{3}} \frac{1}{1+x^2}\, dx$
Now see that $\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}$ is the antiderivative of $\frac{1}{1+x^2}$ but it is not continuous at 1 so we cant use leibnitz rule but then what we can do is break down the interval into two intervals $[0, 1)$ and $(1, \sqrt{2}]$.

7.2 Problems.

(a) Show that $\int_0^{2\pi} \sin^3 x\, dx = 0$

(b) Show that $\int_{-1}^{1} e^{-x^2}\, dx = 2\int_0^1 e^{-x^2}\, dx$

(c) Show that $\int_0^{\pi} \sin x\, dx = 2$ and remember this!

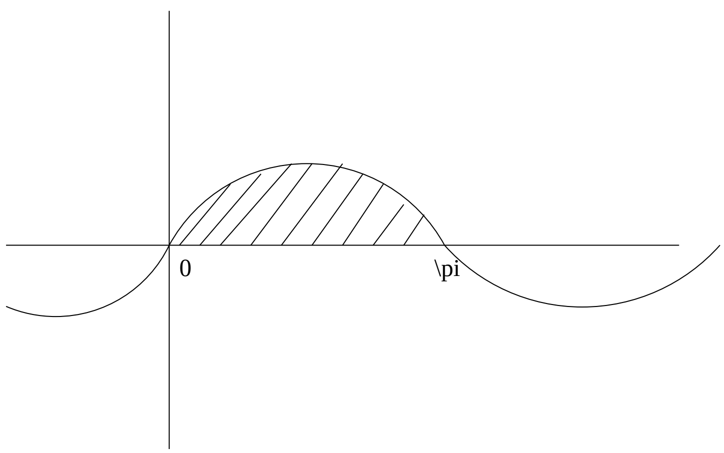


Figure 7.1: area under $\sin(x)$ function in $[0.\pi]$.

(d) Find $\int_0^3 |1-x|\, dx$.

(e) Solve $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\, dx$

(f)

7.3 Properties of Definite Integrals

(a) In definite the variable of integration is the dummy variable i.e.$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

(b) Reversing the limits: $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

(c) If $f(x) \leq \phi(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \leq \int_a^b \phi(x)\,dx$$

(d) A special case is, if $f(x) \geq 0$ then

$$\int_a^b f(x)\,dx \geq 0$$

Example 7.3.1. $0 < x^2 < x < 1$ then

$$0 \leq \int_0^1 x^2\,dx \leq \int_0^1 x\,dx \leq 1$$

Note whether there should be $\leq$ *or* $<$ above.

(e) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Example 7.3.2.

$$\int_1^2 |x|\,dx = \int_1^2 |3-x|\,dx$$

(f) Special case of the above formula
$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(g) From above, $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)\,dx|$ use that $-|x| \leq x \leq |x|$

Example 7.3.3. The function $f(x)$ may be below x-axis also. So the left hand side gives the area under curve which would include the negative area too. So the total area may be less than the absolute..

$$|\int_{-1}^1 x\,dx| \leq \int_{-1}^1 |x|\,dx$$

(h) Note here that the point c might be a point of discontinuity. This is called *piece-wise integration.* So by integrating in small pieces keeping the point of discontinuity we are obeying the definition of Definite Integration. Also note that let the point c be such that $a < c < b$ since we might know how the function behaves in this interval but sometimes in solving problems we might for sake of simplifying the problem take the point c outside the interval (a, b) which might result the inclusion of a point of discontinuity which might lie in the interval (b,c). So beware..!

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(i) $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$

Can be got by first using the piece-wise integration in the interval $(0, a/2)$ and $(a/2, a)$. Then use the substitution $x = a - t$ and using the reversing the integral limits the second property above. This kind of simplification is useful if the function $f(x)$ is symmetric about $x = a/2$ line giving the below short way of seeing.

$$\int_0^a f(x)\,dx = \begin{cases} 2\int_0^{a/2} f(x)\,dx \text{ if } f(a-x) = f(x) \\ 0 \text{ if } f(a-x) = -f(x) \end{cases}$$

Example 7.3.4. $\int_0^\pi \sin(x)\,dx$ and $\int_0^\pi \cos(x)\,dx$

Can be solved since the function $\sin(x)$ is behaving as $f(x) = f(a-x)$ and in the second problem $\cos(x)$ behaves as $f(x) = -f(a-x)$

(j) If the function $f(x)$ repeats itself after period $a/2$ i.e. $f(x) = f(a/2 + x)$ then the following form might be useful.

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a/2 + x)\,dx$$

use the piece-wise integration and then substitute $x = a/2 + t$.

Example 7.3.5. $\int_0^{2\pi} \sin(x)\,dx$

Here in the first problem the function $\cos(x)$ behaves as $f(x) = f(\pi + x)$ and second problem $f(x) = -f(\pi + x)$.

(k)

$$\int_{-a}^a f(x)\,dx = \begin{cases} 0 \text{ if the } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx \text{ if the } f(x) \text{ is even} \end{cases}$$

(l) $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ where m and M are the least and the greatest values of the function $f(x)$

(m) Generalised mean value theorem
$\int_a^b f(x)\phi(x)\,dx = f(\xi)\int_a^b \phi(x)\,dx$ where $a < \xi < b$

(n) Leibnitz's rule for the differentiation under the integral sign
If the function $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at $x \in (a, b)$ and $f(t)$ is continuous on $[\phi(a), \phi(b)]$, then

$$\frac{d}{dx}(\int_{\phi(x)}^{\psi(x)} f(t)\,dt) = \frac{d}{dx}\{\psi(x)\}f(\psi(x)) - \frac{d}{dx}\{\phi(x)\}f(\phi(x))$$

Note: Here in Leibnitz's rule note that the integrand function is a function of t alone and doesn't contain x or functions of x, $\phi(x)$ and $\psi(x)$.

Example 7.3.6. On $[0, \frac{\pi}{2}]$, prove that

$$\int_0^{\sin^2(x)} \sin^{-1}(\sqrt{t})\,dt + \int_0^{\cos^2 x} cos^{-1}(\sqrt{t})\,dt = \frac{\pi}{4}$$

Example 7.3.7. Prove that if $f(x) = \frac{1}{2}\int_0^x (x-t)^2 g(t)\,dt$ then $f'(x) = x\int_0^x g(t)\,dt - \int_0^x tg(t)\,dt$
Note that above in the leibnitz rule the integrand is a function of the dummy variable alone. That point is to be noted while solving this problem as here the integrand is a function of the variable x along with the dummy variable.

Example 7.3.8. $\lim_{h\to 0}\int_2^{2+h} \frac{\sqrt{t^2+2}}{h}\,dt$
$\lim_{x\to 1}\int_1^x \frac{\sin\frac{\pi t}{2}}{(x-1)(t^2+1)}\,dt$
Both the problems can be solved if you are able to see that the limit problem is of the form $\frac{0}{0}$ form. And you can apply *L'Hospital* rule using the Leibnitz rule.

(o) Schwarz-Bunyakovsky inequality
If $f(x)$ and $g(x)$ are integrable on (a, b) then we have

$$|\int_a^b f(x)g(x)\,dx| \leq \sqrt{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}$$

(p) Note the following method with the help of a problem.

Example 7.3.9. Evaluate $F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln(x)}\, dx$, where $\alpha \neq -1$ is a parameter.
First differentiate $F(\alpha)$ w.r.t α the differentiation can be carried inside the integral sign differentiate x^α to get $x^\alpha \ln(x)$ which will simplify to give us.
$F'(\alpha) = \frac{1}{\alpha+1} \Rightarrow F(\alpha) = \ln(\alpha + 1)$ hence the result.
Now this result is used to prove

$$\int_0^1 \frac{x-1}{\ln(x)}\, dx = \ln 2$$

.

7.4 Area Bounded by curves

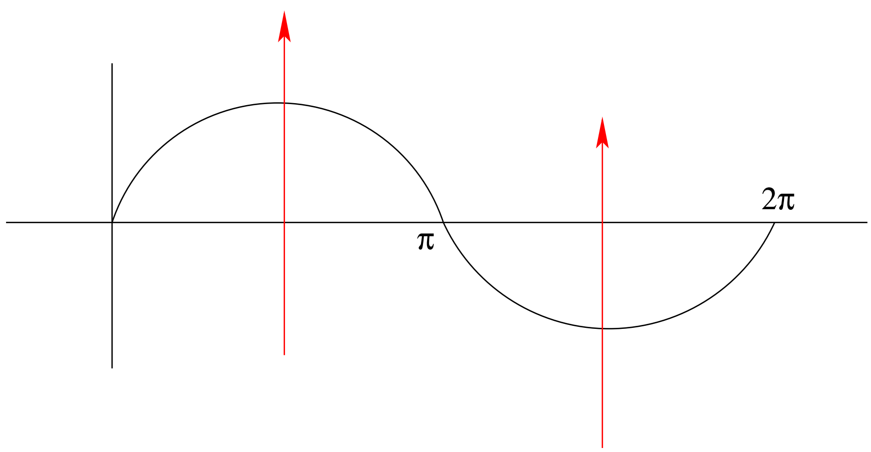
What is the difference between the below two problems,

(a) What is area bounded by the curves $f(x) = \sin(x)$ and x-axis and $x = 0$,$x = 2\pi$ lines.

(b) Evaluate $\int_0^{2\pi} \sin(x)\, dx$

Both the problems are different.The first problem has a non-zero area. The value of the second problem is *zero*.The convention which we would use to solve the area problem is by drawing an arrow in the region whose area need to be calculated. The arrow moves laterally to give the lower and upper limit of x. We cal the function through which the arrow enters the required region as the **initial function** and the function through which it comes out as the **final function**.The direction of the arrow along the positive y-axis is considered to be positive and the lateral movement of the arrow long the positive x-axis is considered to be positive. Now if both this movements are there in a figure then the area evaluated by that integral is positive and other posibilities take place.
For example see the first figure. Here in $[0, \pi]$ the arrow enter the

region through $y = 0$ function and comes out through $y = \sin(x)$ function. So we call $y = 0$ the initial function and the $y = \sin(x)$ the final function. And to calculate the area of this much region using definite integrals we write $\int_0^\pi (\sin(x) - 0)\,dx$,
i.e $\int_0^\pi (\sin(x))\,dx$
Now the remaining area got to be calculated as $\int_\pi^{2\pi} (0 - \sin(x))\,dx$ since now the function have gone upside down so to get the area positive by the arrow convention we got to change the direction of the arrow to upwards.



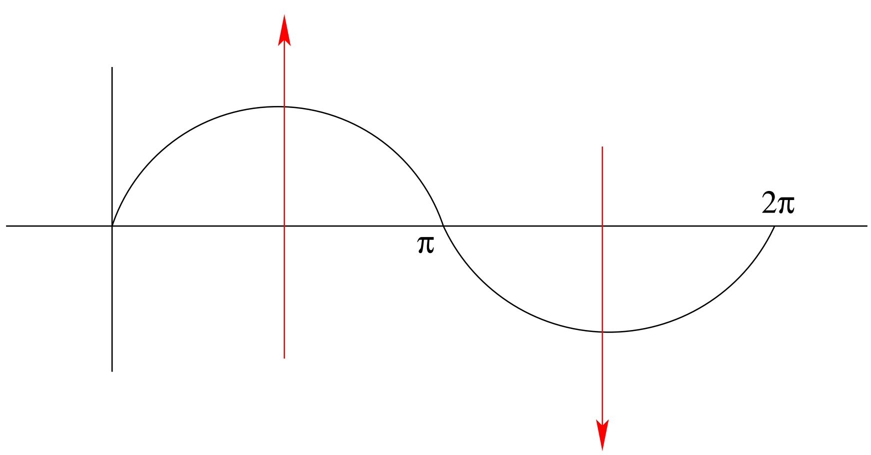


Figure 7.2: The first graph is to the first problem and the second for the second problem, Note the arrow direction in both the graphs.

YOU HAVE TO SOLVE PROBLEMS OF THE TYPE..PAGE 190 PROBLEM 71 I.E PROBLEMS INVOLVING INVERSE TRIGONOMETRIC FUNCTION.OK

Chapter 8

Differential Equations

8.1 Introduction

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a *differential equations.*
If there is only one independent variable then the D.E is called an *ordinary differential equation* or if there are more than one independent variables then the D.E is called *partial differential equation.*

Example 8.1.1. 1.$\frac{dy}{dx} = x + 5$
2.$(y'')^2 + (y')^3 + 3y = x^2$
3.$\frac{\partial^2 z}{\partial x^2} = z + x\frac{\partial z}{\partial y}$
Here above the first two are examples of ordinary D.E and the third is of partial D.E.

What is the need to learn Methods to solve and generate D.E's ?
In any natural process, the variables involved and their rates of change are connected with one another by means of the basic scientific principles that govern the process. When this connection is expressed in mathematical symbols, the result is often a D.E.

Example 8.1.2. Suppose a body of mass m falling freely under gravitational force alone. In this case the only force acting on it is mg, where g is the acceleration due to gravity. If y is the distance down to the body from some fixed height, then its acceleration is $\frac{d^2y}{dt^2}$, and using $F = ma$ we get

$$m\frac{d^2y}{dt^2} = mg$$

or

$$\frac{d^2y}{dt^2} = g$$

So we got a D.E.

8.2 Problem solving

There are two sort of problems involved in Problem solving in D.E.
1. Finding the solution(primitive) to a given D.E.
2. Generating the D.E given the primitive.
Before investigating it and the ways to do this, we need to see first the *order* and *degree* of a differential equation.

Definition 8.2.1. order of a derivative
The number of times the function is differentiated is called the order of the derivative.
For e.g. $\frac{dy}{dx}$ is the first order derivative of y w.r.t x.
And $\frac{d^3y}{dx^3}$ is the third order derivative of y w.r.t x.

Definition 8.2.2. Order of a D.E
Order of a D.E is defined as the order of the *highest* order derivative occuring in the D.E.

Definition 8.2.3. Degree of a D.E
The highest power on the highest order derivative is defined as the degree of the D.E.

Example 8.2.4. (a)

$$\frac{d^2y}{dx^2} + (\frac{d^2y}{dx^2})^3 + 2y = 0$$

Here this D.E is of degree 3 since the highest power on the highest order derivative is 3.

(b)

$$y = x\frac{dy}{dx} + \sqrt{1 + (\frac{dy}{dx})^2}$$

On removing the radicals we get

$$y^2 - 2xy\frac{dy}{dx} + x^2(\frac{dy}{dx})^2 = 1 + (\frac{dy}{dx})^2$$

which has the highest power on the highest order derivative(2) as 2.

Note that the **degree** is defined only when the D.E is a polynomial in derivatives of the dependent variables. see the next example

Example 8.2.5. The degree of the D.E

$$\frac{d^3y}{dx^3} + 2(\frac{dy}{dx})^3 = x\log(\frac{d^2y}{dx^2})$$

For this problem the degree is not defined since its not a polynomial in derivatives of the dependent variable/s.

Note: You should take care of two things while calculating degree of a D.E
1. Get rid of all the denominators in the D.E.
2. Get rid of all the radicals in the differential equations.

Example 8.2.6. Find the degree and order of the given D.E

$$\frac{y}{\frac{dy}{dx}} = x\frac{dy}{dx} + (1 + (\frac{dy}{dx})^2)^{2/3}$$

$$(1 + (\frac{dy}{dx})^2)^{3/2} = \frac{d^2y}{dx^2}$$

Definition 8.2.7. Linear and Non-linear D.E
A D.E which is linear w.r.t the *dependent variable and its derivatives.* i.e. our D.E. should be linear(of degree one) in the the variables $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$.

So any linear D.E is of the form

$$a_0y^{(n)} + a_1y^{(n-1)} + ... + a_{n-1}y^{(1)} + a_ny = p_n$$

where all a_i, $0 \leq i \leq n$ are functions of x.

Example 8.2.8. Are these problems linear or non-linear.
1. $x^2\frac{dy}{dx} + yx^3 = \frac{d^3y}{dx^3}$ is a linear D.E.
2. $y(\frac{dy}{dx}) = x^2$ is not a linear D.E.

Formation of a D.E

You are given an equation relating the independent variable and the dependent variable with a set of **n** *independent* arbitrary constants. Then You follow the steps below
1. Differentiate the equation w.r.t to the independent variable as many times as the number of the arbitrary constants(**n**) in the equation.
2. By this you get n equations with those n arbitrary constants. Now solve this equations to eliminate these n unknown constants.
What are these arbitrary constants in this given equation from which you are going to find the D.E. ?
This arbitrary constants are the parameters which give you a family of curves which all have the same D.E.
For e.g. $x^2+y^2 = c^2$ here the arbitrary constant $c > 0$ will give rise to a family of circles which all would be having the centre as $(0, 0)$ and radius a. So the differential equation of this family is the same.

NOTE: *The order of a D.E is same as the number of* ***independent*** *arbitrary constants in the solution (primitive) of the D.E.* see the next example below.

Example 8.2.9. Find the order of the D.E

(a)

$$y = cx + \sqrt{c}x^2$$

Here note that the arbitrary constants are one and the same.i.e. there is only one arbitrary constant c. So the order is 1. Can you find the D.E of this equation.

(b)
$$y = a\sin(x) + b\cos(x + c)$$

Now what is the order of the corresponding D.E here?? It is 2 since we can solve it to get two *independent* arbitrary constants as..

$$y = (a - b\sin(c))\sin(x) + (b\cos(c)\cos(x)) = k_1\sin(x) + k_2\cos(x).$$

Definition 8.2.10. Particular Solution and General Solution of a D.E
Particular solution of a D.E is defined as the solution without any arbitrary constants i.e. those arbitrary constants are evaluated by given extra condition.
While General solution is the solution to a D.E containing those arbitrary constants.
Since the solution of a D.E is a family of curves (arbitrary constants shows that its a family) the particular solution gives only a particular curve satisfying the extra condition given along with the D.E. The general solution is the whole family of curves.

8.3 Types of problems

We will be seeing the various methods to solve *first order, first degree D.E* of specific forms:

(a) Variable separable

(b) D.Equations of the form $\frac{dy}{dx} = f(ax + by + c)$ (note here f is a function of linear term $ax + by + c$ which finally is converted to *variable separable* form).

(c) D.E of the form containing $xdy - ydx$ or $xdx + ydy$

(d) Homogenous D.E.

(e) Non-homogenous D.E reducible to Homogenous form. $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ which has two subparts
1. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$
2. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(f) Linear D.E. of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.

(g) Bernoulli's equation, which has the form: $\frac{dy}{dx} + Py = Qy^n$

Let us see these methods in detail one by one...
Variable Separable form:
Any D.E of the form $M(x, y)dx + N(x, y)dy = 0$ if you are able to write in the form $f(x)dx = g(y)dy$ then just integrating on both the sides gives you the general solution of the differential equation, then it is said to be in variable separable form.

Example 8.3.1. Solve $\sec^2(x)\tan(y)\,dx + \sec^2(y)\tan(x)\,dy = 0$
The problem is in variable separable form. So we have on rearrangement,
$\frac{\sec^2(x)}{\tan(x)} + \frac{\sec^2(y)}{\tan(y)} = 0$
$\int \frac{\sec^2(x)}{\tan(x)}\,dx + \int \frac{\sec^2(y)}{\tan(y)}\,dy = \ln(c)$,$c > 0$
$\ln|\tan(x)\tan(y)| = \ln(c)$
$\Rightarrow |\tan(x)\tan(y)| = c$

Example 8.3.2. Solve $\frac{dy}{dx} = \sin(x + y) + \sin(x - y)$
As you see it doesn't seem to be in variable separable form but it is.
i.e $\frac{dy}{dx} = 2\sin(x)\cos(y)$
i.e $\int \frac{dy}{\cos(y)} = 2\int \sin(x)\,dx$
i.e $\ln|\sec(y) + \tan(y)| = -2\cos(x) + c$

Note that in the problems that we are going to do hence on, our aim would be reduce that particular form D.E to variable separable form.
D.E Reducible to variable separable forms

(a) *A differential equation of the form* $\frac{dy}{dx} = f(ax + by + c)$ *can be solved by transformation* $t = ax + by + c$

Example 8.3.3. Solve $\frac{dy}{dx} = e^{x+y} + 1$
This problem is not in variable separable form. But we see that the function on the R.H.S is a function of $x + y$.
Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$ So using this we get the D.E in (x, t) as

$$\frac{dt}{dx} - 1 = e^t + 1$$

Now the problem is in variable separable form.

(b) *A differential equation of the form* $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ *and which has* $\frac{a_1}{a_2} = \frac{b_1}{b_2}$
Then use the transformation $t = a_1x + b_1y + c_1$ or $t = a_2x + b_2y + c_3$ whichever is simpler. This transforms the given differential equation to variable separable D.E in (x, t) solve and again resubstitute to get the solution to the original problem in (x, y). **Note here that this form is a special case of the first form done in this list.**

Example 8.3.4. Puting an example is pending.

(c) The next we are going to discuss a *Homogenous D.E*, but before that we need to discuss *homogenous functions*.

Definition 8.3.5. Homogenous functions A function $f(x, y)$ is a homogenous function $\Leftrightarrow f(tx, ty) = t^n f(x, y)$ where t is a variable, then n is the degree of this homogenous function $f(x, y)$.

Example 8.3.6. i. $f(x, y) = x \sin(y) + y \sin(x)$
This function is not a homogenous function since the variable t won't be able to come out of the sin function.

ii. $f(x, y) = x \sin(x/y) + y \sin(y/x)$
This function is a homogenous function of degree 1 since the t inside the sin function wont have any t term but there

would be one t term coming from the x and y term outside the trigonometric functions.

Now we turn to the definition of the *homogenous D.E.*

Definition 8.3.7. A D.E of the form $M(x,y)dx+N(x,y)dy = 0$ or $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ where $M(x,y)$ and $N(x,y)$ are *homogenous functions* of same degree is defined as a Homogenous D.E (H.D.E).

Example 8.3.8. i. $(x^2+y^2)dx + xydy = 0$
Is a homogenous D.E. since both the function x^2+y^2 and xy are homogenous functions of same degree i.e 2. Moreover if we write this D.E in another form

ii. $\frac{dy}{dx} = -\frac{x^2+y^2}{xy}$ then the function $-\frac{x^2+y^2}{xy}$ is a homogenous function of degree Zero.

The method to solve a H.D.E is to use the substitution $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. And on making this substitution we get the final D.E in variable separable form in (v,x) variables.

Example 8.3.9.

$$(1+e^{x/y})dx + e^{x/y}(1-\frac{x}{y})dy = 0$$

This problem is a homogenous D.E. Since both the functions are homogenous functions of degree 0.But then if we make use of $y = vx$ transformation then the new D.E is bit complicated,(try that). But what if we form the new D.E by using the transformation $x = vy$ this would really simplify the new D.E. Let us work that.

$$\frac{dx}{dy} = -\frac{e^{x/y}(1-x/y)}{(1+e^{x/y})}$$

Note here that we have written the D.E in $\frac{dx}{dy}$ form. put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{(1+e^v)}$$

$$y\frac{dv}{dy} = -\frac{v+e^v}{1+e^v}$$
$$\int \frac{1+e^v}{v+e^v}\,dv = -\int \frac{dy}{y} + c$$
$$\ln|v+e^v| + \ln|y| = \ln c'$$
$$(x/y + e^{x/y})y = c'$$

This is the solution to the given D.E.

Note here that the solution to the D.E is a function in implicit form. So whether you have its D.E in $\frac{dy}{dx}$ or $\frac{dx}{dy}$ doesnt matter, since we assume that inverse function exists.

(d) *A D.E of the form* $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ *where* $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ is the next type.
Now this D.E is simplified if the constants terms are somehow removed from the D.E. i.e. the terms c_1,c_2 so that the problem becomes a homogenous D.E. The best way is to use a transformation such that in the new space our D.E is a Homogenous D.E. Let me tell you how.
Substitute $x = X+h$ and $y = Y+h \Rightarrow dx = dX$ and $dy = dY$ where (h,k) is the point where the coordinate system is being moved. We needed c_1 and c_2 to be zero in the new system we can solve the equation $a_1h+b_1k+c_1 = 0$ and $a_2h+b_2k+c_2 = 0$ to make it zero from solving this equations we know now the point where the origin is being moved i.e (h,k).

Example 8.3.10.
$$\frac{dy}{dx} = -\frac{x-1}{y-1}$$

This problem is a variable separable problem. But is of great help if we can also see that it is also in the above form.(i.e. $\frac{dy}{dx} = -\frac{(1)x+(0)y+(-1)}{(0)x+(1)y+(-1)}$).
Now if you solve this problem as a variable separable problem then you get,

$$\int y-1\,dy + \int x-1\,dx = c$$

$$(y-1)^2+(x-1)^2=2c$$

This is equation of the family of the circles centered at $(1,1)$ and radius $\sqrt{2c}$.
Now seeing the same problem with the above type perspective. We will substitute $x = X+1$ and $y = Y+1$ which actually shifts the origin to $(1,1)$ such that now the family of solutions has its center as $(0,0)$. So the D.E has also become homogenous.That's the trick in this method.We use the transformation that simplifies the D.E.
The overall process of solving this kind of D.E is as follows

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2},\ \text{where } \frac{a_1}{a_2}\neq\frac{b_1}{b_2} \Rightarrow \text{ Homogenous D.E } \Rightarrow \text{ Variable separab}$$

Linear Differential Equation of First Order
A D.E of the form

$$\frac{dy}{dx}+yP(x)=Q(x)$$

We see that,

$$\frac{d}{dx}(y\cdot e^{\int P\,dx}) = y\cdot e^{\int P\,dx}\frac{d}{dx}(\int P\,dx)+e^{\int P\,dx}\frac{dy}{dx}=Q\cdot e^{\int P\,dx}$$

Therefore the final solution is,

$$y\cdot e^{\int P\,dx}=\int Q\cdot e^{\int P\,dx}\,dx+c$$

8.4 Some short-cut formulae

(a) $d(\frac{y}{x})=\frac{xdy-ydx}{x^2}$
(b) $d(xy)=xdy+ydx$
(c) $d(\frac{x}{y})=\frac{ydx-xdy}{y^2}$
(d) $d(\log(\frac{y}{x})=\frac{xdy-ydx}{xy}$
(e) $d(\tan^{-1}(\frac{y}{x}))=\frac{xdy-ydx}{x^2+y^2}$
(f) $d(\tan^{-1}(\frac{x}{y}))=\frac{ydx-xdy}{x^2+y^2}$

(g) $d(\frac{1}{2}\log(x^2+y^2)) = \frac{xdx+ydy}{x^2+y^2}$

(h) $\frac{1}{2}d(x^2+y^2) = xdx+ydy$

(i) $d(\log(\frac{x}{y})) = \frac{ydx-xdy}{xy}$

(j) $d(\log(x+y) = \frac{dx+dy}{x+y}$

(k) $d(\log(xy)) = \frac{xdy+ydx}{xy}$

Some D.E can be made exact to use the above formulae by multiplying by a factor called *integrating factor*.

(a) Integrating factor of a homogeneous D.E.
$Mdx+Ndy=0$ is $\frac{1}{Mx+Ny}$ where $Mx+Ny \neq 0$

(b) Integrating factor of D.E of the form
$Mydx+Nxdy=0$ is $\frac{1}{Mx-Ny}$ where $Mx-Ny \neq 0$

Chapter 9

Trigonometry

Chapter 10

Theory of Equations

10.1 Roots of Quadratic Equations

Definition 10.1.1. A Quadratic equation is defined as $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$.

Let α, β be the roots of the given quadratic equation. Then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

And the roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

10.2 Generation of new equation

This is true for any equation. If we have an equation $f(x) = 0$ with the root α then if we need a new equation whose root ϕ is related to α as $\alpha = g(\phi)$ then we get the new equation as $f(g(\phi)) = 0$. This new idea can be extended to quadratic equations also.(provided that function $g(x)$ should be *bijective* Think why??)
This concept can be used to form new quadratic equations if the new roots related to α and β are given.

10.3 Graphical study of properties of the quadratic expression

Let $f(x) = ax^2 + bx + c$ then we discuss the key factors deciding the study of quadratic equations and inequalities.
1. Concavity is decided by the sign of the a
2. Intersection of the quadratic (which is a parabola think! why

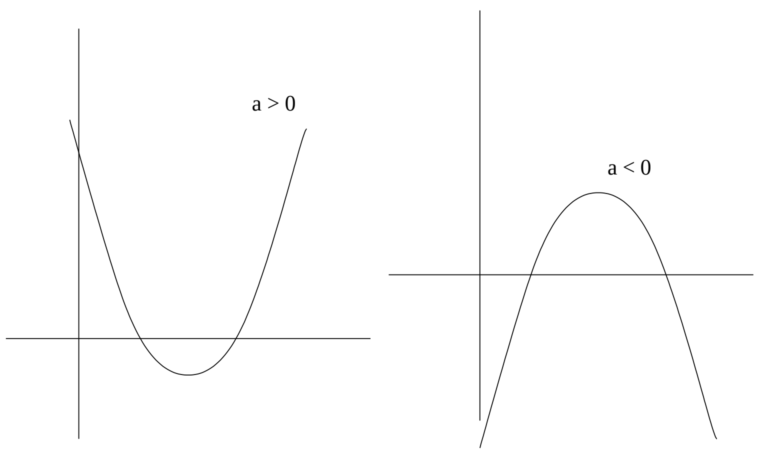


Figure 10.1: Convavity depends on sign of a

this is always upwards or downwards concavity) curve with the x-axis is decided by the sign of the *discriminant* $\Delta = b^2 - 4ac$

(a) If $\boldsymbol{\Delta > 0}$ then the curve will intersect x-axis in **exactly two points**.

(b) If $\boldsymbol{\Delta = 0}$ then the curve will touch the x-axis in **exactly one point**.

(c) If $\boldsymbol{\Delta < 0}$ then the curve will **not** intersect the x-axis

(d)

Chapter 11

Inequalities

11.1 Basic Inequalities

If a, b, c are real numbers then

(a) If $a < b \Rightarrow a + c < b + c$ (note its true whether $c > 0$ or $c < 0$ or $c = 0$

(b) If $a < b \Rightarrow ac < bc$ for $c > 0$ or $ac > bc$ for $c < 0$.

(c) If $a < b$ and $p, q > 0$ then $a^p < b^p$ and $a^{1/q} < b^{1/q}$.

(d) If $0 < a < 1 \Rightarrow 0 < \cdots < a^3 < a^2 < a < 1$
and if $a > 1 \Rightarrow 1 < a < a^2 < \cdots < a^n - 1 < \cdots < \infty$.

11.2 A.M-G.M inequality

There is this heavily used inequality called the **A.M-G.M inequality** If $a, b > 0$ then $\frac{a+b}{2} \geq \sqrt{ab}$
It can also be generalised for n **positive** real numbers $\{a_1, a_2, ..., a_n\}$ then we have $\frac{\sum a_i}{n} \geq (\Pi a_i)^{1/n}$
NOTE A.M-G.M inequality works only for positive real values. And A.M-G.M inequality will be useful when you see symmetry in the question since the inequality is also symmetrical or cyclic..
Can you prove the A.M-G.M inequality with two methods.
Hint: Method-1 is to use $(a - b)^2 \geq 0$ and Method-2 is to use

11.3 Symmetry

The maximum of minimum value (which ever exists) of an expression is attained only when all the terms in the symmetrical expression are equal.

Example 11.3.1. The minimum value of the expression $(a-b)^2$ exists and is 0 when $a = b$.

Refer Hall and Knight.

Chapter 12

Difficult General Problems.

12.1 Logarithms problems (inequations and equations)

(a) What is the difference between a equation and a inequation.(think graphically.)

(b) $(\frac{1}{5})^{\frac{2x+1}{1-x}} > (\frac{1}{5})^{-3}$
solution: $(1,4)$

(c) $(\frac{1}{3})^{\frac{|x+2|}{2-|x|}} > 9$
solution: $(2,6)$

(d) $\sqrt{9^x + 3^x - 2} \geq 9 - 3^x$
solution: $(2, +\infty)$

(e) $2^x + 2^{|x|} \geq 2\sqrt{2}$
solution:$(-\infty, \log_2(\sqrt{2}-1)] \cup (1/2, +\infty)$

(f) $(1/2)^{log_2(x^2-1)} > 1$
solution: $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$

(g) $|x-1|^{\log^2 x - \log x^2} = |x-1|^3$

(h) Find the minimum value of the expression $|\log_a b + \log_b a|$

12.2 Quadratic equations

(a) For what values of $a \in R$ does the equation $ax^2 + x + a - 1 = 0$ possess two distinct real roots x_1 and x_2 satisfying the inequality $|\frac{1}{x_1} - \frac{1}{x_2}| > 1$? Hint: the equation is $f(x) = 0$ then get the equation $f(\frac{1}{x})$ with the roots $\frac{1}{x}$ now the given condition above is $|x_1 - x_2| > 1$ ..solve!!

(b)

12.3 Drawing the graphs

(a) Draw the graphs of $y = x^2$ and $y = x^3$ in the same coordinate system.(i.e. together) what do u learn from it ?

(b) Draw $\log_a(x)$ and a^x for $a > 1$ and $0 < a < 1$

(c) $f(x) = x + \sin(x)$ and $g(x) = x - sin(x)$

(d) $f(x) = x + \frac{1}{x}$ and $g(x) = x - \frac{1}{x}$

(e) $|x| + |y| = 1$ what region does this graph bound ?

(f) Draw the graph of the function $f(x) = x|x|$ where $x \in R$.
Is this function Differentiable at $x = 0$ or continuous at $x = 0$??

12.4 Continuity

(a) The function $f(x) = \frac{1}{x}$ for $x \in R - \{0\}$ is continuous function or not ? Why ?
solution: The function is not defined at $x = 0$. But since the point $x = 0$ removed from the domain of the function we can always draw a tangent at any point in its domain. So the function is differentiable at every point and a differentiable function is continous at every point of the function domain. So this function is continous. We might be misguided by the lame definition that(which is also right) if we are able to draw the graph of the function without raising the pencil then that function is continuous.

(b)

12.5 Probability

(a) If the probability of success on a single experiment with n outcomes is $1/n$.
1. show that the probability that, in m trials there is no success is $(1-\frac{1}{n})^m$.
2. show that if $m = n\ln 2$ then $\lim_{n\to\infty}(1-1/n)^m = \frac{1}{2}$.

(b) My dad heard this story on the radio. At Duke University, two students had received A's in chemistry all semester. But on the night before the final exam, they were partying in another state and didn't get back to Duke until it was over. Their excuse to the professor was that they had a flat tire, and they asked if they could take a make-up test. The professor agreed, wrote out a test and sent the two separate rooms to take it. The first question (on one side of the paper) was worth 5 points, and they answered it easily. Then they flipped the paper over and found the second question, worth 95 points:"which tire was it?" what was the probability that both students would say the same thing?
Solution: 1/4 since here we need to find the probability that both say the same tire no.which are $(1,1),(2,2),(3,3),(4,4)$ so 4/16.

(c) The following question was asked of a class of students. "I was driving to school today, and aone of my tires went flat. which tire do you think it was?" Assume that the two-test takers are randomly chosen from the general population. What is the probability that they will give the right answer to this question.
Solution: should be 1/16 since here the person are randomly chosen then the probability that both would give the right answer is 1/16.

12.6 Theory Questions

(a) What is a theorem??
A statement if is true for a few examples then can we say that we have proved that its a theorem !
What is a converse of a theorem??
If a statement is true say "If ... then ..." then can we say that the theorem is also true in the converse side..??
If a statement is being proved to be theorem.. then can we say its true for a given case...??

(b) Is $\log_a N^r = r \log_a N$ ...? when can we say that the identity is true?

(c)

12.7 Pool of Ist level problems.

Example 12.7.1. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} > \sqrt{n}$ for an arbitrary natural $n \geq 2$. Solution:
We show that $\frac{1}{\sqrt{n+1}} > \sqrt{n+1} - \sqrt{n}$ see how?? adding the above question's L.H.S on both sides and using the inequality true for $k = n$ we get the result true for n+1 also. hence proved by induction.

12.8 Pool of IInd level of problems